Symmetries and Conservation Laws for the Wave Equations of Scalar Statistical Optics

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Abstract. The Lie method and Noether's theorem are applied to the double wave equations for the correlation functions of statistical optics. Generalizations of the deterministic conservation laws are found and seen to correspond to the usual laws in the deterministic limit. The statistically stationary wave equations are shown to contain fewer symmetries than for the nonstationary case, so the corresponding conservation laws differ from the conservation laws of the nonstationary, two-time, wave equations.

1. Introduction

At optical frequencies, the electric field oscillates too rapidly to be measured directly. Instead, detectors produce a signal proportional to the time-averaged intensity of the field. The field itself must often be considered to be stochastic, and most observable quantities are related to the second-order moments of the field: the spectral density, the cross-spectral density, the autocorrelation function, and the cross-correlation function. The cross-spectral density and the cross-correlation function obey double wave equations, the Wolf equations [1]. These correlation functions may thus be propagated without knowledge of the underlying random fields.

A symmetry of an equation is a transformation which does not alter the set of solutions of the equation. In this work, a Lie transformation refers to a continuous infinitesimal transformation, and a Lie symmetry refers to a Lie transformation which is a symmetry of an equation. The symmetry group of a differential equation is the largest group of continuous transformations infinitesimally close to the identity element which leave the set of solutions of the equation unchanged [2]. The symmetry group can be specified by a group of infinitesimal generators which define a Lie algebra [2]. The procedure to find the symmetry group [2, 3, 4] is called the Lie method [5]. Symmetries of an equation are closely related to conservation laws. Noether's theorem [6, 7, 8] provides a method for finding conservation laws of a differential equations arising from a known Lagrangian L and having a known Lie symmetry.

The study of symmetries and conservation laws of the equations of electromagnetics has played an important part in the advancement of physics. For example, special relativity was discovered by considering the symmetries of the equations of electromagnetics [9, 10]. Lorentz observed that Maxwell's equations [11] in free space were invariant upon a rotation involving the coordinates of both position and time [12]. This transformation is now known as a Lorentz transformation. Poincaré and Einstein generalized this concept to Maxwell's equations with currents and charges [13]. This analysis led to an unintuitive explanation for the relative lengths and motion of objects traveling near the speed of light.

Symmetry analysis was also used to identify the relationship between the force exerted by a charged object on another charged object and the force exerted by a current carrying wire on another current carrying wire. Not long after Maxwell's equations were published, Heaviside observed that they are invariant upon the discrete symmetry transformation $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$, and he explored the implications of this duality between the electric and magnetic fields [14]. Larmor and Rainich, however, realized that this relationship is described by a continuous symmetry which is more general than the relationship found by Heaviside [9, 5, 15]. Fushchich and Nikitin [9, 5] generalized this symmetry to a family of related symmetries and invariants for Maxwell's equations with and without sources.

More recently, conservation of energy, which is a direct result of the time translation symmetry of the optical wave equations, has been used to explore variations in the spectrum of light upon propagation. It was observed that the cross-spectral density of light may vary upon propagation even in free space [16, 17, 18, 19, 20], and this observation raised questions about whether or not this change violated conservation of energy in the statistical setting [21]. Energy conservation was shown to be valid and has been studied thoroughly in both scalar statistical [22, 21, 23, 24] and vector statistical [25, 26] descriptions of optics. Aside from being of fundamental scientific interest, implications of energy conservation in scattering of stochastic fields has been shown to have applications in imaging [27, 28, 29]. Energy conservation is just one conservation law of the second order correlations of stochastic fields. Others, including momentum, angular momentum, and their generalizations, may be identified. Many of these conservation laws or invariants may be discovered through the study of the continuous symmetries of the Wolf equations.

There are three main results in this paper. First, the symmetry group of the wave equations of scalar statistical optics is derived. Second, corresponding conservation laws are found using Noether's theorem. Third, changes in the symmetry group of the wave equation that result when the optical signal is assumed to be stationary are discussed. Conservation laws found include the well-studied conservation laws for energy, momentum, and angular momentum as well as conservation laws corresponding to inversion and dilatation symmetries. The general two-time wave equations are found to contain inversion symmetries which are not present in the stationary wave equations. The paper is organized as follows. In this section, the wave equations of scalar statistical optics and the procedure to find the symmetry group of an equation are discussed. In Section 2, the symmetry group of the wave equations of scalar statistical optics is derived. In

Section 2.2, the symmetry group of the wave equations is derived for stationary optical signals. Conservation laws are discussed in Section 3. Conclusions and future directions are discussed in Section 4.

1.1. Equations of Scalar Statistical Optics

In scalar wave optics, light is described by a deterministic complex analytic scalar field, $\varphi(\mathbf{r}, t)$, which obeys a wave equation

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0. \tag{1}$$

The scalar *statistical* description incorporates the effects of random fluctuations of optical sources or random fluctuations introduced when light propagates through the atmosphere [1, 30]. Detectors cannot respond at optical frequencies, and instead time averaged quantities are measured, usually the time averaged second moment of the field. The cross-correlation $\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$, also called mutual coherence, of a random function is defined [30]

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) = \langle\varphi^*(\mathbf{r}_1, t_1)\varphi(\mathbf{r}_2, t_2)\rangle\,, \tag{2}$$

where the angle brackets denote ensemble averaging. In Cartesian coordinates, $\mathbf{r}_1 = x_1\mathbf{a}_{x_1} + y_1\mathbf{a}_{y_1} + z_1\mathbf{a}_{z_1}$ and $\nabla_1 = \partial_{x_1}\mathbf{a}_{x_1} + \partial_{y_1}\mathbf{a}_{y_1} + \partial_{z_1}\mathbf{a}_{z_1}$. The cross-correlation for fields propagating in free space with no sources obeys a pair of wave equations, called the Wolf equations [1],

$$\nabla_\beta^2\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) = \frac{1}{c^2}\frac{\partial^2}{\partial t_\beta^2}\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) \tag{3}$$

for $\beta = 1$ and $\beta = 2$. These wave equations govern the propagation of the cross-correlation.

A random process is stationary if all of its probability densities are symmetric with respect to t through the origin of time, and it is defined to be stationary in the wide sense if all of its second order averages depend on $\tau = t_1 - t_2$, the difference in times, but not t_1 and t_2 separately [30]. The cross-correlation of a stationary random process is written

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle\varphi^*(\mathbf{r}_1, t+\tau)\varphi(\mathbf{r}_2, t)\rangle\,. \tag{4}$$

If the random process is also ergodic, the ensemble average may be replaced by an time average.

1.2. Lie Method

Consider a set of equations with a set of independent variables χ^i labeled by i, for example the six components of $\mathbf{r}_1$ and $\mathbf{r}_2$ and the time coordinates t_1 and t_2, and one dependent complex variable Γ. A continuous transformation may be denoted $\chi^i \to A(\theta)\chi^i$ and $\Gamma \to A(\theta)\Gamma$ where θ is a continuous parameter. For a transformation infinitesimally close to the identity, the mapping $A(\theta)$ may be expressed by a Taylor expansion [4]

$$A(\theta) = 1 + \theta U + \frac{1}{2}\theta^2 U^2 + ... \, . \tag{5}$$

The quantity U is called an infinitesimal generator, and it has the form

$$U = \eta_\Gamma \partial_\Gamma + \eta_{\Gamma^*} \partial_{\Gamma^*} + \sum_i \xi^i \partial_{\chi^i}. \tag{6}$$

The functions ξ^i, η_Γ, and η_{Γ^*} may depend on the independent and dependent variables. By exponentiation, the corresponding transformation may be found from an infinitesimal generator. The transformation of each independent variable is given by [4]

$$\chi^i \to e^{\theta U} \chi^i \tag{7}$$

which can be written as $\chi^i \to \chi^i \left(1 + \theta \xi^i\right)$ in the limit of small θ. The transformation of the dependent variable is given by

$$\Gamma \to e^{\theta U} \Gamma \tag{8}$$

which can be written as $\Gamma \to \Gamma \left(1 + \theta \eta_\Gamma\right)$ in the limit of small θ, and similarly, $\eta_{\Gamma^*} = \eta_\Gamma^*$ represents the transformation of the complex conjugate of Γ. The infinitesimal generators corresponding to all symmetries form a group [4]. For example, Eq. (1), contains a translation symmetry for each independent variable. Time translation, which can be denoted $t \to t + \theta$, is a symmetry because the set of solutions of Eq. (1) is unaltered when t is shifted by the constant parameter θ. The corresponding infinitesimal generator is $U = \partial_t$. The infinitesimal transformation can be recovered by exponentiation the infinitesimal generator $e^{\theta \partial_t} t = t + \theta$.

The procedure to find the symmetry group of an equation is based on the idea that a symmetry does not alter the set of solutions of the equation. When an equation is acted upon by a symmetry described by the infinitesimal generator U, the set of solutions of the transformed and original equations are the same. Additionally, all higher derivatives of the set of solutions of the transformed and original equations are unchanged. The n-th prolongation of the generator $\mathrm{pr}^{(n)}U$ is a generalization of the generator U that incorporates the transformations of the derivatives of the dependent variables in a manner consistent with the transformation of the dependent variable. All infinitesimal symmetries of an equation $\Delta = 0$ must satisfy

$$\mathrm{pr}^{(n)} U \Delta = 0, \tag{9}$$

for all positive integers n [2]. This infinitesimal criterion of invariance, also called the symmetry criterion, can often be solved for the generators corresponding to all Lie symmetries of the original equation, and this procedure can be directly generalized to find the symmetry group of sets of differential equations [2].

The prolongation of an infinitesimal generator in the form of Eq. (6), with one dependent complex variable, is given by the expression [2]

$$\mathrm{pr}^{(n)}U = U + \sum_J \eta_\Gamma^J \frac{\partial}{\partial\left(\Gamma\right)_J} + \eta_{\Gamma^*}^J \frac{\partial}{\partial\left(\Gamma^*\right)_J}, \tag{10}$$

where the index J runs over the set composed of the independent variables, products of two independent variables, and all products of up to n independent variables, and here the subscript J denotes partial derivative. For example $\Gamma_{xy} = \frac{\partial^2\Gamma}{\partial x \partial y}$. Functions η_Γ^J are defined [2] by

$$\eta_\Gamma^J = \frac{d}{dJ}\left(\eta_\Gamma - \sum_i \xi^i \frac{\partial\Gamma}{\partial\chi^i}\right) + \sum_i \xi^i \frac{\partial}{\partial\chi^i}\left(\Gamma\right)_J, \tag{11}$$

where the index i runs over the independent variables. For a differential equation of order k, the k-th prolongation $\mathrm{pr}^{(k)}U$ acting on the equation will be equal to all prolongations of order greater than k. Thus, for a second order differential equation, only the second prolongation is needed.

The symmetry group of Eq. (1) was first identified in [31], and the symmetry group of Maxwell's equations was first identified in [32]. Both [31] and [32] were written before Lie's work on continuous symmetries. The Lie method is used to find the symmetry group of the wave equation with one dependent and three independent variables in [2]. In references [9, 5], the Lie method is used to find the symmetry group, and symmetries of Maxwell's equations are discussed in more detail for waves propagating both in free space and in the presence of sources. The symmetry group of Eqs. (3) is studied here.

Not all symmetries of an equation are Lie symmetries or can be found by the Lie method. Discrete transformations cannot be represented by infinitesimal generators and are not Lie symmetries. Certain discrete, as opposed to continuous, symmetries of the wave equations of scalar and vector statistical optics have been discussed in [30, 33, 25, 34, 35]. Continuous symmetries which can be described by generators in the form of Eq. (6) but for which the functions ξ^i and η_Γ depend on derivatives of the variables are called dynamical symmetries [2, 3], and they will not be considered in this paper. Nongeometrical symmetries can be described by infinitesimal generators but are not continuous [9, 5]. For example, a nongeometrical symmetry may involve taking a Fourier transform, performing an infinitesimal translation, or other continuous transformation, in the frequency domain, then taking an inverse Fourier transform. Nongeometrical symmetries also will not be considered here.

1.3. Noether's Theorem

Noether's theorem provides a systematic means to obtain conservation laws from Lie symmetries of two types: variational symmetries and divergence symmetries. Not all conservation laws can be found from Noether's theorem [5], yet in many cases it provides a direct procedure for finding conservation laws. A Lie symmetry is called variational if and only if its infinitesimal generator satisfies [2],

$$\mathrm{pr}^{(n)}UL + \sum_i L\frac{\partial \xi^i}{\partial \chi^i} = 0, \tag{12}$$

for an equation or set of equations with independent variables χ^i and Lagrangian L. A Lie symmetry is called a divergence symmetry if it satisfies [2],

$$\mathrm{pr}^{(n)}UL + \sum_i L\frac{\partial \xi^i}{\partial \chi^i} = \sum_i \frac{\partial B_i}{\partial \chi^i}, \tag{13}$$

for some vector $\mathbf{B} = \sum_i B_i \mathbf{a}_i$ where $\mathbf{a}_i$ denotes the unit vector in the direction of increasing χ^i. Noether's theorem states that for both variational and divergence symmetries, corresponding conservation laws can be found. For a variational symmetry, the conservation law has the form [2]

$$\sum_i \frac{\partial P_i}{\partial \chi^i} = 0, \tag{14}$$

for a vector $\mathbf{P} = \sum_i P_i \mathbf{a}_i$ and for a divergence symmetry, the conservation law has the form

$$\sum_i \frac{\partial}{\partial \chi^i}\left(P_i - B_i\right) = 0. \tag{15}$$

In both cases, for an equation with one dependent complex variable, the vector $\mathbf{P}$ is given by [2]

$$P_i = \eta_\Gamma \frac{\partial L}{\partial\left(\frac{\partial \Gamma}{\partial \chi^i}\right)} + \eta_{\Gamma^*}\frac{\partial L}{\partial\left(\frac{\partial \Gamma^*}{\partial \chi^i}\right)} + \xi^i L - \sum_j \left[\xi^j \frac{\partial \Gamma}{\partial \chi^j}\frac{\partial L}{\partial\left(\frac{\partial \Gamma}{\partial \chi^i}\right)} + \xi^j \frac{\partial \Gamma^*}{\partial \chi^j}\frac{\partial L}{\partial\left(\frac{\partial \Gamma^*}{\partial \chi^i}\right)}\right]. \tag{16}$$

Noether's theorem has been used to find the conservation laws corresponding to all of the types of Lie symmetries of Eq. (1) and of Maxwell's equations [9, 5, 2, 36, 37]. Also, certain conservation laws for the sets of wave equations for the cross-correlations in scalar statistical optics [22, 30, 21] and vector statistical optics [25, 35, 26] have been studied though not systematically nor exhaustively.

2. Lie Analysis

In this section, the symmetry group of the two-time wave equations for the cross-correlation, defined by Eq. (2), is derived using the Lie method described above. Subsequently, the symmetry group of the differential equations of the cross-correlation for the stationary case, defined in Eq. (4), is derived. It is seen that a reduction in the number of independent variables results in a reduction of symmetries of the differential equations.

2.1. Two-Time Wave Equations

The pair of equations for the cross-correlation of an optical signal described in scalar-statistical optics, given by Eq. (3), can be written as

$$\Delta_\beta = \nabla^2_\beta \Gamma - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t^2_\beta} = 0 \tag{17}$$

where $\beta = 1$ and 2. These equations have one dependent complex variable Γ and eight independent variables which are given by $\mathbf{r}_1$, $\mathbf{r}_2$, t_1, and t_2. In this section, the Lie method is used to find the infinitesimal generators of the symmetry group of the set of equations. The generators have the form of Eq. (6). The symmetries of Eqs. (17) are derived without considering boundary conditions. For a symmetry to be present in an application, it must be present in both the underlying equations and the boundary conditions. For this reason, the symmetries present in any application will be a subset of the symmetries found here.

The symmetry criteria for Eqs. (17) can be found using Eq. (9),

$$\mathrm{pr}^{(n)} U \Delta_\beta = \eta_\Gamma^{x_\beta x_\beta} + \eta_\Gamma^{y_\beta y_\beta} + \eta_\Gamma^{z_\beta z_\beta} - \frac{1}{c^2}\eta_\Gamma^{t_\beta t_\beta} = 0. \tag{18}$$

The functions $\eta_\Gamma^{x_\beta x_\beta}$, $\eta_\Gamma^{y_\beta y_\beta}$, $\eta_\Gamma^{z_\beta z_\beta}$, and $\eta_\Gamma^{t_\beta t_\beta}$ are defined by Eq. (11), and a related equation holds for η_{Γ^*}. Using the Lie method, it is possible to determine all functions ξ^i and η_Γ that satisfy the criteria of Eqs. (18). The symmetry criteria can only be satisfied when $\frac{\partial \xi^i}{\partial \Gamma} = \frac{\partial^2 \eta_\Gamma}{\partial \Gamma^2} = 0$. Thus, the symmetry criteria may be written as

$$\begin{aligned} &\left(\nabla^2_\beta \eta_\Gamma - \frac{1}{c^2}\frac{\partial^2 \eta_\Gamma}{\partial t^2_\beta}\right) + 2\left(\nabla_\beta \Gamma \cdot \nabla_\beta \frac{\partial \eta_\Gamma}{\partial \Gamma} - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_\beta}\frac{\partial \eta_\Gamma}{\partial \Gamma \partial t_\beta} + \nabla_\beta \Gamma^* \cdot \nabla_\beta \frac{\partial \eta_\Gamma}{\partial \Gamma^*} - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_\beta}\frac{\partial \eta_\Gamma}{\partial \Gamma^* \partial t_\beta}\right) \\ &\quad - \sum_i \left[2\left(\nabla_\beta \xi^i \cdot \nabla_\beta \frac{\partial \Gamma}{\partial \chi^i} - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t_\beta \partial \chi^i}\frac{\partial \xi^i}{\partial t_\beta}\right) + \frac{\partial \Gamma}{\partial \chi^i}\left(\nabla^2_\beta \xi^i - \frac{1}{c^2}\frac{\partial^2 \xi^i}{\partial t^2_\beta}\right)\right] = 0 \end{aligned} \tag{19}$$

where the index i runs over the independent variables.

First consider transformations for which all $\xi^i = 0$. In this case, Eqs. (19) are of the form of Eq. (17), so the symmetry criteria can be satisfied by $\eta_\Gamma = \Gamma$. All linear equations contain the symmetry $\Gamma \to \Gamma + \theta\Gamma$ which corresponds to the infinitesimal generator $U = \Gamma\partial_\Gamma + \Gamma^*\partial_{\Gamma^*}$ [2]. The symmetry criteria can also be satisfied when $\eta_\Gamma = \gamma$ where γ is any solution of Eqs. (17) written as a function of the independent variables. Additionally, the symmetry criteria can be satisfied by $U = \Gamma^*\partial_\Gamma + \Gamma\partial_{\Gamma^*}$.

Next consider transformations for which $\eta_\Gamma = \eta_{\Gamma^*} = 0$. From the term $\nabla_\beta \xi^i \cdot \nabla_\beta \frac{\partial \Gamma}{\partial \chi^i} - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t_\beta \partial \chi^i}\frac{\partial \xi^i}{\partial t_\beta}$, the functions ξ^{x_β}, ξ^{y_β}, ξ^{z_β}, and ξ^{t_β} cannot depend on the variables $x_{\bar\beta}$, $y_{\bar\beta}$, $z_{\bar\beta}$, and $t_{\bar\beta}$ where $\bar\beta = 1$ when $\beta = 2$ and where $\bar\beta = 2$ when $\beta = 1$. Furthermore, the following relationships must hold,

$$\frac{\partial \xi^{y_\beta}}{\partial x_\beta} + \frac{\partial \xi^{x_\beta}}{\partial y_\beta} = \frac{\partial \xi^{z_\beta}}{\partial x_\beta} + \frac{\partial \xi^{x_\beta}}{\partial z_\beta} = \frac{\partial \xi^{y_\beta}}{\partial z_\beta} + \frac{\partial \xi^{z_\beta}}{\partial y_\beta} = 0, \tag{20}$$

$$\frac{\partial \xi^{t_\beta}}{\partial x_\beta} - \frac{1}{c^2}\frac{\partial \xi^{x_\beta}}{\partial t_\beta} = \frac{\partial \xi^{t_\beta}}{\partial y_\beta} - \frac{1}{c^2}\frac{\partial \xi^{y_\beta}}{\partial t_\beta} = \frac{\partial \xi^{t_\beta}}{\partial z_\beta} - \frac{1}{c^2}\frac{\partial \xi^{z_\beta}}{\partial t_\beta} = 0, \tag{21}$$

and

$$\frac{\partial \xi^{x_\beta}}{\partial x_\beta} = \frac{\partial \xi^{y_\beta}}{\partial y_\beta} = \frac{\partial \xi^{z_\beta}}{\partial z_\beta} = \frac{\partial \xi^{t_\beta}}{\partial t_\beta}. \tag{22}$$

Using Eqs. (20) and (22),

$$\frac{\partial^2 \xi^{x_\beta}}{\partial y_\beta^2} = -\frac{\partial^2 \xi^{y_\beta}}{\partial x_\beta \partial y_\beta} = -\frac{\partial^2 \xi^{x_\beta}}{\partial x_\beta^2}. \tag{23}$$

Similar relationships hold for the other independent variables, and

$$\frac{\partial^2 \xi^i}{\partial x_\beta^2} = \frac{\partial^2 \xi^i}{\partial y_\beta^2} = \frac{\partial^2 \xi^i}{\partial z_\beta^2} = \frac{\partial^2 \xi^i}{\partial t_\beta^2} = 0. \tag{24}$$

Since Eqs. (17) describe a pair of equations, the above conditions apply when $\beta = 1$ and when $\beta = 2$. Using these conditions, the infinitesimal generators with $\eta_\Gamma = 0$ may be found. The infinitesimal generators of Eqs. (17) include eight translations,

$$U = \partial_{\chi^i}, \tag{25}$$

where χ^i stands for each independent variable. The generators include six rotations,

$$U = x_\beta \partial_{y_\beta} - y_\beta \partial_{x_\beta}, \quad U = x_\beta \partial_{z_\beta} - z_\beta \partial_{x_\beta}, \quad U = z_\beta \partial_{y_\beta} - y_\beta \partial_{z_\beta}, \tag{26}$$

and six hyperbolic rotations,

$$U = t_\beta \partial_{x_\beta} + \frac{1}{c^2} x_\beta \partial_{t_\beta}, \quad U = t_\beta \partial_{y_\beta} + \frac{1}{c^2} y_\beta \partial_{t_\beta}, \quad U = t_\beta \partial_{z_\beta} + \frac{1}{c^2} z_\beta \partial_{t_\beta}. \tag{27}$$

The generators also include two dilatations,

$$U = \mathbf{r}_\beta \cdot \nabla_\beta + t_\beta \partial_{t_\beta}. \tag{28}$$

These generators include rotations amongst the coordinates of $\mathbf{r}_\beta$ and the coordinates of $\mathbf{r}_{\bar{\beta}}$ separately but not rotations between the coordinates of $\mathbf{r}_\beta$ and $\mathbf{r}_{\bar{\beta}}$ together.

Only cases where some or all of the ξ^i along with η_Γ and η_{Γ^*} are nonzero remain to be considered. The following generators satisfy the symmetry criteria.

$$U = \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right)\partial_{x_\beta} + 2x_\beta y_\beta \partial_{y_\beta} + 2x_\beta z_\beta \partial_{z_\beta} + 2x_\beta t_\beta \partial_{t_\beta} - 2\Gamma x_\beta \partial_\Gamma - 2\Gamma^* x_\beta \partial_{\Gamma^*}, \tag{29}$$

$$U = 2x_\beta y_\beta \partial_{x_\beta} + \left(-x_\beta^2 + y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right)\partial_{y_\beta} + 2y_\beta z_\beta \partial_{z_\beta} + 2y_\beta t_\beta \partial_{t_\beta} - 2\Gamma y_\beta \partial_\Gamma - 2\Gamma^* y_\beta \partial_{\Gamma^*}, \tag{30}$$

$$U = 2x_\beta z_\beta \partial_{x_\beta} + 2y_\beta z_\beta \partial_{y_\beta} + \left(-x_\beta^2 - y_\beta^2 + z_\beta^2 + c^2 t_\beta^2\right)\partial_{z_\beta} + 2z_\beta t_\beta \partial_{t_\beta} - 2\Gamma z_\beta \partial_\Gamma - 2\Gamma^* z_\beta \partial_{\Gamma^*}, \tag{31}$$

$$U = 2x_\beta t_\beta \partial_{x_\beta} + 2y_\beta t_\beta \partial_{y_\beta} + 2z_\beta t_\beta \partial_{z_\beta} + \frac{1}{c^2}\left(x_\beta^2 + y_\beta^2 + z_\beta^2 + c^2 t_\beta^2\right)\partial_{t_\beta} - 2\Gamma t_\beta \partial_\Gamma - 2\Gamma^* t_\beta \partial_{\Gamma^*}. \tag{32}$$

The symmetries corresponding to these infinitesimal generators are are known as inversion symmetries [2]. Equations (29) - (32) specify eight inversion generators for β=1 and $\beta = 2$. Inversion symmetries are members of the conformal group [5, 31]. By definition [38], a conformal transformation is a transformation which preserves the angles between the coordinate axes.

It may be seen that the infinitesimal generators for the two-time wave equations, Eqs. (17), are the generators for the wave equation of the deterministic field, Eq. (1), repeated over both sets of four coordinates x_β, y_β, z_β, and t_β. In the next section, it is seen that this is not the case for the wave equations for the cross-correlations of the stationary fields.

2.2. Lie Analysis of Wave Equations of Stationary Statistical Optics

Most physical optical systems which are described in scalar statistical optics are well modeled by assuming the signals are stationary and ergodic [33]. The wave equations for the cross-correlation function of a stationary random processes can be written with seven, as opposed to eight independent variables. All stationary random processes contain time translation symmetries for both t_1 and t_2 which allows the cross-correlation to be written as a function of $\tau = t_1 - t_2$ as opposed to the time coordinates individually. The cross-correlation of stationary random processes described in scalar statistical optics obeys the pair of wave equations

$$\Lambda_\beta = \nabla_\beta^2 \Gamma - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial \tau^2} = 0 \tag{33}$$

for $\beta = 1$ and $\beta = 2$. In this section, the symmetry group of Eqs. (33) is derived. These equations involve one dependent complex variable Γ and seven independent variables given by $\mathbf{r}_1$, $\mathbf{r}_2$, and τ.

The symmetry group of Eqs. (33) can be found using the Lie method. Infinitesimal generators have the form of Eq. (6). The symmetry criteria for Eqs. (33) are

$$\mathrm{pr}^{(n)} U \Lambda_\beta = \eta_\Gamma^{x_\beta x_\beta} + \eta_\Gamma^{y_\beta y_\beta} + \eta_\Gamma^{z_\beta z_\beta} - \frac{1}{c^2}\eta_\Gamma^{\tau\tau} = 0. \tag{34}$$

The functions $\eta_\Gamma^{x_\beta x_\beta}$, $\eta_\Gamma^{y_\beta y_\beta}$, $\eta_\Gamma^{z_\beta z_\beta}$, and $\eta_\Gamma^{\tau\tau}$ are defined by Eq. (11). Unlike in the two-time case, the two symmetry criteria, with $\beta = 1$ and 2, are coupled because they both involve the function $\eta_\Gamma^{\tau\tau}$. As in the two-time case, and for the same reasons, to satisfy the symmetry criteria, both $\frac{\partial \xi^i}{\partial \Gamma}$ and $\frac{\partial^2 \eta_\Gamma}{\partial \Gamma^2}$ must be zero. Thus, the symmetry criteria can be written as

$$\begin{aligned}\left(\nabla_\beta^2 \Gamma - \frac{1}{c^2}\frac{\partial^2 \eta_\Gamma}{\partial \tau^2}\right) + 2\left(\nabla_\beta \Gamma \cdot \nabla_\beta \frac{\partial \eta_\Gamma}{\partial \Gamma} - \frac{1}{c^2}\frac{\partial \Gamma}{\partial \tau}\frac{\partial \eta_\Gamma}{\partial \Gamma \partial \tau} + \nabla_\beta \Gamma^* \cdot \nabla_\beta \frac{\partial \eta_\Gamma}{\partial \Gamma^*} - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial \tau}\frac{\partial \eta_\Gamma}{\partial \Gamma^* \partial \tau}\right) \\ - \sum_i \left[2\left(\nabla_\beta \xi^i \cdot \nabla_\beta \frac{\partial \Gamma}{\partial \chi^i} - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial \tau \partial \chi^i}\frac{\partial \xi^i}{\partial \tau}\right) + \frac{\partial \Gamma}{\partial \chi^i}\left(\nabla_\beta^2 \xi^i - \frac{1}{c^2}\frac{\partial^2 \xi^i}{\partial \tau^2}\right)\right] = 0.\end{aligned} \tag{35}$$

As in the case of the two-time wave equations, some symmetries involve only transformations of the dependent variable, and for these symmetries, all ξ^i are zero. Equations (33) are linear, so $\eta_\Gamma = \Gamma$ satisfies the symmetry criteria. The symmetry criteria also can be satisfied by $\eta = \gamma$ where γ is any solution of Eq. (33) written as a function of the independent variables. Additionally, the symmetry criteria is satisfied by $U = \Gamma \partial_{\Gamma^*} + \Gamma^* \partial_\Gamma$.

Equations (33) also contains some symmetries, with $\eta_\Gamma = 0$, which involve transformations of the independent variables but not Γ. For the term $\left(\nabla_\beta \xi^i \cdot \nabla_\beta \frac{\partial \Gamma}{\partial \chi^i} - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial \tau \partial \chi^i}\frac{\partial \xi^i}{\partial \tau}\right)$ to satisfy the symmetry criteria, the functions ξ^{x_β}, ξ^{y_β}, and ξ^{z_β} cannot depend on $x_{\bar{\beta}}$, $y_{\bar{\beta}}$, and $z_{\bar{\beta}}$, so Eq. (20) again holds. However, here Eqs. (21) and (22) are replaced by the expressions

$$\frac{\partial \xi^\tau}{\partial x_\beta} - \frac{1}{c^2}\frac{\partial \xi^{x_\beta}}{\partial \tau} = \frac{\partial \xi^\tau}{\partial y_\beta} - \frac{1}{c^2}\frac{\partial \xi^{y_\beta}}{\partial \tau} = \frac{\partial \xi^\tau}{\partial z_\beta} - \frac{1}{c^2}\frac{\partial \xi^{z_\beta}}{\partial \tau} = 0, \tag{36}$$

and

$$\frac{\partial \xi^{x_\beta}}{\partial x_\beta} = \frac{\partial \xi^{y_\beta}}{\partial y_\beta} = \frac{\partial \xi^{z_\beta}}{\partial z_\beta} = \frac{\partial \xi^\tau}{\partial \tau}. \tag{37}$$

For the last term of Eq. (35) to satisfy the symmetry criteria, all ξ^i must be at most linear in the independent variables.

Similar to Eqs. (17), the resulting symmetries of Eqs. (33), with $\eta_\Gamma = 0$, can be classified as translations, rotations, and dilatations. The infinitesimal generators of Eqs. (33) include seven translations which are given by Eq. (25) where χ^i ranges over seven rather than eight independent variables. They also include six rotation generators given by Eq. (26) and three hyperbolic rotation generators, which involve τ rather than t_1 and t_2, represented by

$$U = \tau\partial_{x_\beta} + \tau\partial_{x_{\bar{\beta}}} + \frac{1}{c^2}\left(x_\beta + x_{\bar{\beta}}\right)\partial_\tau, \tag{38}$$

$$U = \tau\partial_{y_\beta} + \tau\partial_{y_{\bar{\beta}}} + \frac{1}{c^2}\left(y_\beta + y_{\bar{\beta}}\right)\partial_\tau, \tag{39}$$

and

$$U = \tau\partial_{z_\beta} + \tau\partial_{z_{\bar{\beta}}} + \frac{1}{c^2}\left(z_\beta + z_{\bar{\beta}}\right)\partial_\tau. \tag{40}$$

They also include one dilatation generator

$$U = \sum_{i=1}^{7} \chi^i \partial_{\chi^i} \tag{41}$$

rather than the two generators given by Eq. (28).

Equations (33) contain no symmetries for which η_Γ and at least one of the ξ^i are nonzero. It is not possible to find choices of η_Γ and ξ^i for which terms of Eq. (35) are individually nonzero yet sum to zero. Consider the possibility $\eta_\Gamma = -2\Gamma x_\beta$ for which the second term of Eq. (35) is nonzero. This choice satisfies the condition that η_Γ must be at most linear in Γ. Other choices of the independent variable could be made. As in the two-time case, from the term $\left(\nabla_\beta \xi^i \cdot \nabla_\beta \frac{\partial\Gamma}{\partial\chi^i} - \frac{1}{c^2}\frac{\partial^2\Gamma}{\partial\tau\partial\chi^i}\frac{\partial\xi^i}{\partial\tau}\right)$, generators must satisfy Eqs. (20), (36), and (37). Thus, both $\frac{\partial\xi^{x_\beta}}{\partial x_\beta} = \frac{\partial\xi^\tau}{\partial\tau}$ and $\frac{\partial\xi^{x_{\bar{\beta}}}}{\partial x_{\bar{\beta}}} = \frac{\partial\xi^\tau}{\partial\tau}$ must be satisfied, and from Eqs. (36) and (37),

$$\frac{\partial^2\xi^{x_\beta}}{\partial\tau^2} = c^2\frac{\partial^2\xi^\tau}{\partial x_\beta^2}. \tag{42}$$

However, Eq. (42) cannot be satisfied because ξ^{x_β} cannot depend on $x_{\bar{\beta}}$. Thus, no inversion symmetries can be found.

Both Eqs. (17) and (33) contain many of the same types of symmetries, including translations, rotations, hyperbolic rotations, dilatations, and symmetries due to linearity. However, since Eqs. (33) involve fewer independent variables, fewer infinitesimal generators are needed span the Lie algebra than for Eqs. (17). Unlike Eqs. (17), Eqs. (33) do not contain inversion symmetries.

3. Conservation Laws

In this section, Noether's theorem is applied to Lie symmetries derived in Section 2 to find conservation laws for the cross-correlation function for both the two-time and stationary cases. In each case, the conservation law for the deterministic case is provided to place the generalization to the stochastic case in context. It is seen that the conservation laws derived for the two-time wave equations encompass and are generalizations of the conservation laws for the deterministic fields in the sense that the conservation laws for the cross-correlation function reduce to the conservation laws for the deterministic field in the deterministic limit. Moreover, conservation laws for deterministic fields may be seen to have multiple generalizations in the context of the cross-correlation function for stochastic fields.

3.1. Two-Time Wave Equations

The Lagrangians for Eqs. (17) are

$$L_\beta = -\left|\nabla_\beta \Gamma\right|^2 + \frac{1}{c^2}\left|\frac{\partial \Gamma}{\partial t_\beta}\right|^2, \tag{43}$$

for $\beta = 1$ and $\beta = 2$. To determine if Noether's theorem is applicable, the first prolongation of a generator acting on the Lagrangians is needed in applying Eqs. (12),

$$\mathrm{pr}^{(1)}UL_\beta = -\frac{\partial \Gamma^*}{\partial x_\beta}\eta_\Gamma^{x_\beta} - \frac{\partial \Gamma^*}{\partial y_\beta}\eta_\Gamma^{y_\beta} - \frac{\partial \Gamma^*}{\partial z_\beta}\eta_\Gamma^{z_\beta} + \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_\beta}\eta_\Gamma^{t_\beta} - \frac{\partial \Gamma}{\partial x_\beta}\eta_{\Gamma^*}^{x_\beta} - \frac{\partial \Gamma}{\partial y_\beta}\eta_{\Gamma^*}^{y_\beta} - \frac{\partial \Gamma}{\partial z_\beta}\eta_{\Gamma^*}^{z_\beta} + \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_\beta}\eta_{\Gamma^*}^{t_\beta}. \tag{44}$$

The functions $\eta_\Gamma^{x_\beta}$, $\eta_{\Gamma^*}^{x_\beta}$, ... are defined by Eq. (11).

The translation symmetries are variational because the corresponding infinitesimal generators satisfy Eq. (12). For deterministic scalar fields, the conserved quantity associated with time translation invariance is called energy Eq. (12). The wave equation for the scalar deterministic field, Eq. (1), contains the energy conservation law in the form of Eq. (14) with

$$\mathbf{P} = \frac{\partial \varphi^*}{\partial t}\nabla\varphi + \frac{\partial \varphi}{\partial t}\nabla\varphi^* + \left(L - \frac{2}{c^2}\left|\frac{\partial \varphi}{\partial t}\right|^2\right)\mathbf{a}_t \tag{45}$$

where L is the Lagrangian of the deterministic wave equation

$$L = -\left|\nabla\varphi\right|^2 + \frac{1}{c^2}\left|\frac{\partial \varphi}{\partial t}\right|^2. \tag{46}$$

For the deterministic case, the $\mathbf{a}_t$ component of $\mathbf{P}$ is referred to as the density of the conserved quantity while the remaining components are referred to as the flux density vector. The energy density is defined as,

$$H(r,t) = |\nabla\varphi|^2 + \frac{1}{c^2}\left|\frac{\partial\varphi}{\partial t}\right|^2, \tag{47}$$

and energy flux density vector is defined as

$$\mathbf{F}(\mathbf{r},t) = -\left(\frac{\partial\varphi^*}{\partial t}\nabla\varphi + \frac{\partial\varphi}{\partial t}\nabla\varphi^*\right). \tag{48}$$

The conservation law given by Eqs. (14) and (45) may be cast in the usual form [30]

$$\frac{\partial H}{\partial t} + \nabla\cdot\mathbf{F} = 0 \tag{49}$$

The conservation law expressed by Eq. (45) may be derived using Noether's theorem, the Lagrangian L, and the single generator for time translation symmetry of Eq. (1). For the statistical case, there are two time coordinates, and the cross-correlation of the field is described by two Lagrangians. Thus, there are four generalizations of Eq. (45). The first two generalizations may be found by applying Noether's theorem with $U = \partial_{t_\beta}$ and Lagrangian L_β, where $\beta = 1$ and $\beta = 2$. Conservation laws of the form of Eq. (14) result with

$$\mathbf{P} = \frac{\partial\Gamma}{\partial t_\beta}\left(\nabla_\beta\Gamma^*\right) + \frac{\partial\Gamma^*}{\partial t_\beta}\left(\nabla_\beta\Gamma\right) + \left(L_\beta - \frac{2}{c^2}\left|\frac{\partial\Gamma}{\partial t_\beta}\right|^2\right)\mathbf{a}_{t_\beta}. \tag{50}$$

Taking $U = \partial_{t_\beta}$ with Lagrangian $L_{\bar\beta}$, conservation laws of the form of Eq. (14) result with

$$\mathbf{P} = \frac{\partial\Gamma}{\partial t_\beta}\left(\nabla_{\bar\beta}\Gamma^*\right) + \frac{\partial\Gamma^*}{\partial t_\beta}\left(\nabla_{\bar\beta}\Gamma\right) + L_{\bar\beta}\mathbf{a}_{t_\beta} - \frac{1}{c^2}\left(\frac{\partial\Gamma}{\partial t_\beta}\frac{\partial\Gamma^*}{\partial t_{\bar\beta}} + \frac{\partial\Gamma^*}{\partial t_\beta}\frac{\partial\Gamma}{\partial t_{\bar\beta}}\right)\mathbf{a}_{t_{\bar\beta}}. \tag{51}$$

The generalizations of energy conservation expressed in Eqs. (50) and (51) become redundant and reduce to Eq. (45) in the deterministic limit where $\Gamma\left(\mathbf{r}_1,t_1;\mathbf{r}_2t_2\right) = \varphi^*\left(\mathbf{r}_1,t_1\right)\varphi\left(\mathbf{r}_2,t_2\right),$ as might be expected.

For deterministic scalar fields, the conserved quantity associated with spatial translation symmetry is called momentum [2, 37]. The wave equation for the scalar deterministic field, Eq. (1), contains the momentum conservation law corresponding to translation symmetry of the x coordinate in the form of Eq. (14) with

$$\mathbf{P} = L\mathbf{a}_x + \frac{\partial\varphi^*}{\partial x}\nabla\varphi + \frac{\partial\varphi}{\partial x}\nabla\varphi^* - \frac{1}{c^2}\left(\frac{\partial\varphi^*}{\partial x}\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial\varphi^*}{\partial t}\right)\mathbf{a}_t. \tag{52}$$

The deterministic wave equation also contains similar conservation laws corresponding to translation symmetry of the y and z coordinates. The t component of Eq. (52) forms the x component of the so-called momentum density vector [37],

$$\mathcal{P} = \frac{1}{c^2}\frac{\partial \varphi^*}{\partial t}\nabla\varphi + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\nabla\varphi^* \tag{53}$$

commonly encountered in deterministic wave optics. For the statistical case, there are six spatial coordinates, and Eqs. (17) are described by two Lagrangians. Thus, there are twelve generalizations of the conservation laws of the form of Eq. (52). The conservation law for translation symmetry along x_β, given by generator $U = \partial_{x_\beta}$, and Lagrangian L_β , for example, has the form of Eq. (16) with

$$\mathbf{P} = L_\beta \mathbf{a}_{x_\beta} + \frac{\partial \Gamma}{\partial x_\beta}\left(\nabla_\beta \Gamma^*\right) + \frac{\partial \Gamma^*}{\partial x_\beta}\left(\nabla_\beta \Gamma\right) - \frac{1}{c^2}\left(\frac{\partial \Gamma^*}{\partial t_\beta}\frac{\partial \Gamma}{\partial x_\beta} + \frac{\partial \Gamma}{\partial t_\beta}\frac{\partial \Gamma^*}{\partial x_\beta}\right)\mathbf{a}_{t_\beta}. \tag{54}$$

A vector formed by the $\mathbf{a}_{t_\beta}$ components of Eq. (54) along with the $\mathbf{a}_{t_\beta}$ components from similar expressions found using L_β with $U = \partial_{y_\beta}$ and $U = \partial_{z_\beta}$ may be defined as a generalization of the momentum density vector,

$$\mathcal{P}_{\beta\beta} = \frac{1}{c^2}\left(\frac{\partial \Gamma^*}{\partial t_\beta}\nabla_\beta \Gamma + \frac{\partial \Gamma}{\partial t_\beta}\nabla_\beta \Gamma^*\right). \tag{55}$$

Another set of conservation laws generalizing momentum conservation may be obtained by taking translations along the β coordinate together with the Lagrangian $L_{\bar{\beta}}$. For example, for translation symmetry along x_β and Lagrangian $L_{\bar{\beta}}$ has the form of Eq. (14) with

$$\mathbf{P} = \frac{\partial \Gamma}{\partial x_\beta}\left(\nabla_{\bar{\beta}} \Gamma^*\right) + \frac{\partial \Gamma^*}{\partial x_\beta}\left(\nabla_{\bar{\beta}} \Gamma\right) + L_{\bar{\beta}} \mathbf{a}_{x_\beta} - \frac{1}{c^2}\left(\frac{\partial \Gamma}{\partial x_\beta}\frac{\partial \Gamma^*}{\partial t_{\bar{\beta}}} + \frac{\partial \Gamma^*}{\partial x_\beta}\frac{\partial \Gamma}{\partial t_{\bar{\beta}}}\right)\mathbf{a}_{t_{\bar{\beta}}}, \tag{56}$$

suggesting the definition of another momentum density vector

$$\mathcal{P}_{\beta\bar{\beta}} = \frac{1}{c^2}\left(\frac{\partial \Gamma^*}{\partial t_{\bar{\beta}}}\nabla_\beta \Gamma + \frac{\partial \Gamma}{\partial t_{\bar{\beta}}}\nabla_\beta \Gamma^*\right). \tag{57}$$

As in energy conservation, momentum conservation of Eqs. (54) and (56) reduces to Eq. (52) in the deterministic limit where $\Gamma(\mathbf{r}_1, t_1; \mathbf{r}_2 t_2) = \varphi^*(\mathbf{r}_1, t_1)\,\varphi(\mathbf{r}_2, t_2)$.

The rotation and hyperbolic rotation generators all correspond to variational symmetries and all satisfy Eq. (12). In the deterministic case, the conserved quantity associated with rotational symmetry is known as angular momentum [2]. The generator $U = x\partial_y - y\partial_x$ represents rotation about the $\mathbf{a}_z$ axis and provides a conservation law in the form of Eq. (14) with

$$\mathbf{P} = \left(-y\mathbf{a}_x + x\mathbf{a}_y\right)L + \left(-y\frac{\partial \varphi}{\partial x} + x\frac{\partial \varphi}{\partial y}\right)\left(\nabla\varphi^* - \frac{1}{c^2}\frac{\partial \varphi^*}{\partial t}\mathbf{a}_t\right) + \left(-y\frac{\partial \varphi^*}{\partial x} + x\frac{\partial \varphi^*}{\partial y}\right)\left(\nabla\varphi - \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\mathbf{a}_t\right). \tag{58}$$

The $\mathbf{a}_t$ component of Eq. (58) is

$$\mathbf{P}\cdot\mathbf{a}_t = \mathbf{r}\times\mathcal{P}\cdot\mathbf{a}_z, \tag{59}$$

where $\mathcal{P}$ is defined by Eq. (53). Together the $\mathbf{a}_t$ components of the conservation laws corresponding to rotations about the $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ axes are defined [37] as the angular momentum density vector $\mathbf{M} = \mathbf{r}\times\mathcal{P}$. The remaining components of the conservation laws form the angular momentum flux density. For the statistical case, the conserved quantities found by applying Noether's theorem using the rotation generators correspond to generalized angular momenta. The conservation law obtained by applying Noether's theorem with the generator $U = x_\beta\partial_{y_\beta} - y_\beta\partial_{x_\beta}$, corresponding to a rotation about the $\mathbf{a}_{z_\beta}$ axis, and Lagrangian L_β is of the form of Eq. (14) with

$$\begin{aligned}\mathbf{P} &= -y_\beta L_\beta \mathbf{a}_{x_\beta} + x_\beta L_\beta \mathbf{a}_{y_\beta} + \left(-y_\beta\frac{\partial\Gamma}{\partial x_\beta} + x_\beta\frac{\partial\Gamma}{\partial y_\beta}\right)\left(\nabla_\beta\Gamma^* - \frac{1}{c^2}\frac{\partial\Gamma^*}{\partial t_\beta}\mathbf{a}_{t_\beta}\right) \\ &+ \left(-y_\beta\frac{\partial\Gamma^*}{\partial x_\beta} + x_\beta\frac{\partial\Gamma^*}{\partial y_\beta}\right)\left(\nabla_\beta\Gamma - \frac{1}{c^2}\frac{\partial\Gamma}{\partial t_\beta}\mathbf{a}_{t_\beta}\right).\end{aligned} \tag{60}$$

A generalization of the angular momentum density vector may be defined by $M_{\beta\beta\beta} = \mathbf{r}_\beta\times\mathcal{P}_{\beta\beta}$ with $\mathcal{P}_{\beta\beta}$ given by Eq. (55). The $\mathbf{a}_{t_\beta}$ component of Eq. (60) may be seen to be identical with the $\mathbf{a}_{z_\beta}$ component of the generalized angular momentum density $M_{\beta\beta\beta}$. Application of Noether's theorem with the generators of the rotations about the other axes yield corresponding vectors $\mathbf{P}$ whose $\mathbf{a}_{t_\beta}$ components are the respective components of $M_{\beta\beta\beta}$. This process may be repeated for the $\beta = 1$ and $\beta = 2$ cases resulting in six generalizations of the usual three angular momentum conservation laws. An additional six conservation laws may be found using the Lagrangian $L_{\bar\beta}$ in which appear an alternative generalization of the angular momentum density vector $M_{\beta\beta\bar\beta} = \mathbf{r}_\beta\times\mathcal{P}_{\beta\bar\beta}$ with $\mathcal{P}_{\beta\bar\beta}$ given by Eq. (57). For example, the conservation law found using the generator $U = x_\beta\partial_{y_\beta} - y_\beta\partial_{x_\beta}$ and Lagrangian $L_{\bar\beta}$ is of the of Eq. (14) with

$$\begin{aligned}\mathbf{P} &= -y_\beta L_{\bar\beta} \mathbf{a}_{x_\beta} + x_\beta L_{\bar\beta} \mathbf{a}_{y_\beta} + \left(-y_\beta\frac{\partial\Gamma}{\partial x_\beta} + x_\beta\frac{\partial\Gamma}{\partial y_\beta}\right)\left(\nabla_{\bar\beta}\Gamma^* - \frac{1}{c^2}\frac{\partial\Gamma^*}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right) \\ &+ \left(-y_\beta\frac{\partial\Gamma^*}{\partial x_\beta} + x_\beta\frac{\partial\Gamma^*}{\partial y_\beta}\right)\left(\nabla_{\bar\beta}\Gamma - \frac{1}{c^2}\frac{\partial\Gamma}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right).\end{aligned} \tag{61}$$

As for energy conservation, Eq. (61) reduces to Eq. (60) in the deterministic limit where $\Gamma(\mathbf{r}_1, t_1; \mathbf{r}_2 t_2) = \varphi^*(\mathbf{r}_1, t_1)\,\varphi(\mathbf{r}_2, t_2)$.

The hyperbolic rotations also correspond to variational symmetries for both the deterministic and statistical wave equations, and conservation laws may be found using Noether's theorem. Using Eq. (7)

it may be seen that the hyperbolic rotation generators of the deterministic wave equation correspond to the Lorentz transformations of special relativity. For example, for the deterministic wave equations, using the generator $U = t\partial_x + \frac{1}{c^2}x\partial_t$ and the Lagrangian L, a conservation law in the form of Eq. (14) is found with

$$\mathbf{P} = \left(t\mathbf{a}_x + \frac{1}{c^2}x\mathbf{a}_t\right)L + \left(t\frac{\partial\varphi}{\partial x} + \frac{1}{c^2}x\frac{\partial\varphi}{\partial t}\right)\left(\nabla\varphi^* - \frac{1}{c^2}\frac{\partial\varphi^*}{\partial t}\mathbf{a}_t\right) + \left(t\frac{\partial\varphi^*}{\partial x} + \frac{1}{c^2}x\frac{\partial\varphi^*}{\partial t}\right)\left(\nabla\varphi - \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\mathbf{a}_t\right). \tag{62}$$

For the statistical wave equations, the generator $U = t_\beta\partial_{x_\beta} + \frac{1}{c^2}x_\beta\partial_{t_\beta}$ along with Lagrangian L_β corresponds to a conservation law in the form of Eq. (14) with

$$\begin{aligned}\mathbf{P} = &\left(t_\beta\mathbf{a}_{x_\beta} + \frac{1}{c^2}x_\beta\mathbf{a}_{t_\beta}\right)L_\beta + \left(t_\beta\frac{\partial\Gamma}{\partial x_\beta} + \frac{1}{c^2}x_\beta\frac{\partial\Gamma}{\partial t_\beta}\right)\left(\nabla_\beta\Gamma^* - \frac{1}{c^2}\frac{\partial\Gamma^*}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\\ &+\left(t_\beta\frac{\partial\Gamma^*}{\partial x_\beta} + \frac{1}{c^2}x_\beta\frac{\partial\Gamma^*}{\partial t_\beta}\right)\left(\nabla_\beta\Gamma - \frac{1}{c^2}\frac{\partial\Gamma}{\partial t_\beta}\mathbf{a}_{t_\beta}\right).\end{aligned} \tag{63}$$

In the deterministic case with three hyperbolic rotation generators and one Lagrangian, three independent conservation laws are found. In the statistical case with six hyperbolic rotation generators and two Lagrangians, twelve independent conservation laws are found.

Both the deterministic wave equation, Eq. (1), and the statistical wave equations, Eqs. (17), contain dilatation symmetries and corresponding conservation laws. Using Eq. (7), it may be seen that the dilatation generators describe the continuous symmetry where all of the variables are scaled by the same constant, $\chi^i \to \chi^i e^\theta$, for constant θ. Applications of dilatation symmetry are discussed in [31, 39]. In the deterministic case, the single Lagrangian along with the generator

$$U = \mathbf{r}\cdot\nabla + t\partial_t - \varphi\partial_\varphi - \varphi^*\partial_{\varphi^*}, \tag{64}$$

which is a linear combination of the dilatation and linearity generators [2], corresponds to a conservation law in the form of Eq. (14) with

$$\mathbf{P} = (\mathbf{r} + t\mathbf{a}_t)L + \left(\nabla\varphi^* - \frac{1}{c^2}\frac{\partial\varphi^*}{\partial t}\mathbf{a}_t\right)\left(\varphi + \mathbf{r}\cdot\nabla\varphi + t\frac{\partial\varphi}{\partial t}\right) + \left(\nabla\varphi - \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\mathbf{a}_t\right)\left(\varphi^* + \mathbf{r}\cdot\nabla\varphi^* + t\frac{\partial\varphi^*}{\partial t}\right). \tag{65}$$

As above, the $\mathbf{a}_t$ component of Eq. (65) may be considered the density while the remaining components may be considered the flux density vector. Similarly, for the statistical case of Eqs. (17), two linearly independent conservation laws may be found using the generator

$$U = \mathbf{r}_\beta \cdot \nabla_\beta + t_\beta \partial_{t_\beta} - \Gamma \partial_\Gamma - \Gamma^* \partial_{\Gamma^*}, \tag{66}$$

along with the two Lagrangians. For this generator and Lagrangian L_β, Eqs. (17) contain a conservation law in the form Eq. (14) with

$$\begin{aligned}\mathbf{P} = &\left(\mathbf{r}_\beta + t_\beta \mathbf{a}_{t_\beta}\right) L_\beta + \left(\nabla_\beta \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\left(\Gamma + \mathbf{r}_\beta \cdot \nabla_\beta \Gamma + t_\beta \frac{\partial \Gamma}{\partial t_\beta}\right) \\ &+ \left(\nabla_\beta \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\left(\Gamma^* + \mathbf{r}_\beta \cdot \nabla_\beta \Gamma^* + t_\beta \frac{\partial \Gamma^*}{\partial t_\beta}\right).\end{aligned} \tag{67}$$

Also, the generator of Eq. (66) along with $L_{\bar\beta}$ corresponds to a conservation law in the form of Eq. (14) with

$$\begin{aligned}\mathbf{P} = &\left(\mathbf{r}_\beta + t_\beta \mathbf{a}_{t_\beta}\right) L_{\bar\beta} + \left(\nabla_{\bar\beta} \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right)\left(2\Gamma + \mathbf{r}_\beta \cdot \nabla_\beta \Gamma + t_\beta \frac{\partial \Gamma}{\partial t_\beta}\right) \\ &+ \left(\nabla_{\bar\beta} \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right)\left(2\Gamma^* + \mathbf{r}_\beta \cdot \nabla_\beta \Gamma^* + t_\beta \frac{\partial \Gamma^*}{\partial t_\beta}\right).\end{aligned} \tag{68}$$

In the deterministic limit where $\Gamma(\mathbf{r}_1, t_1; \mathbf{r}_2 t_2) = \varphi^*(\mathbf{r}_1, t_1)\,\varphi(\mathbf{r}_2, t_2)$, the conservation law specified by Eq. (67) reduces to the conservation law of the deterministic field due to the dilatation symmetry.

The deterministic wave equation, Eq. (1), contains inversion symmetries. With four inversion generators and a single Lagrangian, four linearly independent conservation laws may be found. For example the inversion generator of the deterministic wave equation,

$$U = \left(x^2 - y^2 - z^2 + c^2 t^2\right)\partial_x + 2xy\partial_y + 2xz\partial_z + 2tx\partial_t - 2\varphi x \partial_\varphi - 2\varphi^* x \partial_{\varphi^*}, \tag{69}$$

corresponds to a divergence symmetry. Using Noether's theorem, a conservation law may be found in the form of Eq. (15) with

$$\mathbf{B} = 2\left|\varphi\right|^2 \mathbf{a}_x, \tag{70}$$

and

$$\begin{aligned}\mathbf{P} = &\,2xL\left(y\mathbf{a}_y + z\mathbf{a}_z + t\mathbf{a}_t\right) + \left(x^2 - y^2 - z^2 + c^2 t^2\right) L\mathbf{a}_x \\ &+ \left(x^2 - y^2 - z^2 + c^2t^2\right)\frac{\partial \varphi}{\partial x}\left(\nabla\varphi^* - \frac{1}{c^2}\frac{\partial \varphi^*}{\partial t}\mathbf{a}_t\right) + 2x\left(\nabla\varphi^* - \frac{1}{c^2}\frac{\partial \varphi^*}{\partial t}\mathbf{a}_t\right)\left(\varphi + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z} + t\frac{\partial\varphi}{\partial t}\right) \\ &+ \left(x^2 - y^2 - z^2 + c^2t^2\right)\frac{\partial \varphi^*}{\partial x}\left(\nabla\varphi - \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\mathbf{a}_t\right) + 2x\left(\nabla\varphi - \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\mathbf{a}_t\right)\left(\varphi^* + y\frac{\partial\varphi^*}{\partial y} + z\frac{\partial\varphi^*}{\partial z} + t\frac{\partial\varphi^*}{\partial t}\right).\end{aligned} \tag{71}$$

In the statistical case, there are eight inversion generators of the statistical wave equations along with two Lagrangians, so sixteen linearly independent conservation laws may be found for Eqs. (17). The inversion generators correspond to divergence symmetries because they satisfy Eq. (13). As an example, consider the inversion generator given in Eq. (29) and Lagrangian L_β. This generator corresponds to a divergence symmetry,

$$\mathrm{pr}^{(1)}UL_\beta + L_\beta \sum_i \frac{\partial \xi^i}{\partial \chi^i} = 2\Gamma \frac{\partial \Gamma^*}{\partial x_\beta} + 2\Gamma^* \frac{\partial \Gamma}{\partial x_\beta}, \tag{72}$$

with

$$\mathbf{B} = 2\left|\Gamma\right|^2 \mathbf{a}_{x_\beta}. \tag{73}$$

This generator leads to a conservation law in the form of Eq. (15) with $\mathbf{P}$ specified by Eq. (16),

$$\begin{aligned}\mathbf{P} &= 2x_\beta L_\beta \left(y_\beta \mathbf{a}_{y_\beta} + z_\beta \mathbf{a}_{z_\beta} + t_\beta \mathbf{a}_{t_\beta}\right) + \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right) L_\beta \mathbf{a}_{x_\beta} \\ &+ \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right) \frac{\partial \Gamma}{\partial x_\beta} \left(\nabla_\beta \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_\beta}\mathbf{a}_{t_\beta}\right) + 2x_\beta \left(\nabla_\beta \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\left(\Gamma + y_\beta \frac{\partial \Gamma}{\partial y_\beta} + z_\beta \frac{\partial \Gamma}{\partial z_\beta} + t_\beta \frac{\partial \Gamma}{\partial t_\beta}\right) \\ &+ \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right) \frac{\partial \Gamma^*}{\partial x_\beta} \left(\nabla_\beta \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_\beta}\mathbf{a}_{t_\beta}\right) + 2x_\beta \left(\nabla_\beta \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\left(\Gamma^* + y_\beta \frac{\partial \Gamma^*}{\partial y_\beta} + z_\beta \frac{\partial \Gamma^*}{\partial z_\beta} + t_\beta \frac{\partial \Gamma^*}{\partial t_\beta}\right).\end{aligned} \tag{74}$$

The generator of Eq. (29) along with $L_{\bar\beta}$ also corresponds to a divergence symmetry. The associated conservation law is in the form of Eq. (15) with

$$\mathbf{B} = 2x_\beta \left(\Gamma^* \nabla_{\bar\beta}\Gamma + \Gamma \nabla_{\bar\beta}\Gamma^*\right) - \mathbf{a}_{t_{\bar\beta}} \frac{2}{c^2} x_\beta \left(\Gamma^* \frac{\partial \Gamma}{\partial t_{\bar\beta}} + \Gamma \frac{\partial \Gamma^*}{\partial t_{\bar\beta}}\right), \tag{75}$$

and

$$\begin{aligned}\mathbf{P} &= 2x_\beta L_{\bar\beta} \left(y_\beta \mathbf{a}_{y_\beta} + z_\beta \mathbf{a}_{z_\beta} + t_\beta \mathbf{a}_{t_\beta}\right) + \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right) L_{\bar\beta} \mathbf{a}_{x_\beta} \\ &+ \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right) \frac{\partial \Gamma}{\partial x_\beta} \left(\nabla_{\bar\beta} \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right) + 2x_\beta \left(\nabla_{\bar\beta} \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right)\left(\Gamma + y_\beta \frac{\partial \Gamma}{\partial y_\beta} + z_\beta \frac{\partial \Gamma}{\partial z_\beta} + t_\beta \frac{\partial \Gamma}{\partial t_\beta}\right) \\ &+ \left(x_\beta^2 - y_\beta^2 - z_\beta^2 + c^2 t_\beta^2\right) \frac{\partial \Gamma^*}{\partial x_\beta} \left(\nabla_{\bar\beta} \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right) + 2x_\beta \left(\nabla_{\bar\beta} \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_{\bar\beta}}\mathbf{a}_{t_{\bar\beta}}\right)\left(\Gamma^* + y_\beta \frac{\partial \Gamma^*}{\partial y_\beta} + z_\beta \frac{\partial \Gamma^*}{\partial z_\beta} + t_\beta \frac{\partial \Gamma^*}{\partial t_\beta}\right).\end{aligned} \tag{76}$$

As above, the $\mathbf{a}_t$ component of Eq. (71) is the called density of the conserved quantity while the remaining terms are called the flux density vector. Similarly, the $\mathbf{a}_{t_\beta}$ component of Eq. (74) is the generalized density while the remaining terms are a generalization of the flux density vector of the conserved quantity.

3.2. Stationary Wave Equations

Noether's theorem may also be used to find conservation laws for the wave equations for the cross-correlation of stationary stochastic fields. The Lagrangians of Eq. (33) are

$$L_{\beta} = - \left| \nabla_{\beta} \Gamma \right|^2 + \frac{1}{c^2} \left| \frac{\partial \Gamma}{\partial \tau} \right|^2 . \tag{77}$$

Using Noether's theorem and translation generators, conservation laws can be found corresponding to conservation of generalized energy and generalized momentum density. With seven translation generators and two Lagrangians, fourteen linearly independent conservation laws can be found. In the stationary case, because there is only one time coordinate, there are only two generalizations of conservation of energy. With the generator $U = \partial_{\tau}$ and Lagrangian L_{β}, the generalized energy conservation law has the form of Eq. (14) with

$$\mathbf{P} = \frac{\partial \Gamma}{\partial \tau} \left(\nabla_{\beta} \Gamma^{*} \right) + \frac{\partial \Gamma^{*}}{\partial \tau} \left(\nabla_{\beta} \Gamma \right) + \left(L_{\beta} - \frac{2}{c^2} \left| \frac{\partial \Gamma}{\partial \tau} \right|^2 \right) \mathbf{a}_{\tau} . \tag{78}$$

For the conservation law of Eq. (78), the $\mathbf{a}_{\tau}$ component $\left(L_{\beta} - \frac{2}{c^2} \left| \frac{\partial \Gamma}{\partial \tau} \right|^2 \right)$ may be considered the generalized energy density, and the remaining components may be considered the generalized energy flux density vector. This conservation law is related to the conservation law of Eqs. (50) and (51) for the two-time wave equations. It may be seen by making the change of variables $\tau = t_1 - t_2$ and $T = \frac{t_1 + t_2}{2}$ and enforcing $\frac{\partial \Gamma}{\partial T} = 0$, essentially a restatement of stationarity, that Eqs. (50) and (51) become redundant and equivalent to Eq. (78). However, since the cross-correlation function in the stationary case depends on only one time coordinate, there is no direct path to a deterministic limit for Eq. (78). That is, Eq. (78) expresses a conservation law unique to the statistically stationary setting.

Conservation laws may also be found for the spatial translation generators for Eqs. (33) as for the two-time wave equations. The momentum conservation laws again take the form of Eq. (14) with $\mathbf{P}$ as given in Eqs. (54) and (56) but with $t_{\beta} \to \tau$.

Conservation laws of the rotation and hyperbolic rotation generators in the stationary and two-time cases are also closely related. For the generator $U = x_{\beta} \partial_{y_{\beta}} - y_{\beta} \partial_{x_{\beta}}$, for example, the conservation law from Noether's theorem and Lagrangian L_{β} has the form of Eq. (14) with $\mathbf{P}$ as given in Eq. (60) with $t_{\beta} \to \tau$. Also, a conservation law may be found using Lagrangian $L_{\bar{\beta}}$ which has the form of Eq. (14) with $\mathbf{P}$ as given in Eq. (61) again with $t_{\beta} \to \tau$. The hyperbolic rotation generator of Eq. (38) corresponds to a conservation law in the form of Eq. (14) with

$$\mathbf{P} = \left[\tau \mathbf{a}_{x_\beta} + \tau \mathbf{a}_{x_{\bar{\beta}}} + \frac{1}{c^2}\left(x_\beta + x_{\bar{\beta}}\right)\mathbf{a}_\tau\right] L_\beta \tag{79}$$

$$+ \left(\nabla_\beta \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial \tau}\mathbf{a}_\tau\right)\left[\tau \frac{\partial \Gamma}{\partial x_\beta} + \tau \frac{\partial \Gamma}{\partial x_{\bar{\beta}}} + \frac{1}{c^2}\left(x_\beta + x_{\bar{\beta}}\right)\frac{\partial \Gamma}{\partial \tau}\right]$$

$$+ \left(\nabla_\beta \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial \tau}\mathbf{a}_\tau\right)\left[\tau \frac{\partial \Gamma^*}{\partial x_\beta} + \tau \frac{\partial \Gamma^*}{\partial x_{\bar{\beta}}} + \frac{1}{c^2}\left(x_\beta + x_{\bar{\beta}}\right)\frac{\partial \Gamma^*}{\partial \tau}\right]. \tag{80}$$

The conservation laws for the hyperbolic rotation generators in the stationary case results from two conservation laws for the hyperbolic rotation generators in the two-time case. The conservation law of Eq. (79) of the stationary case may be found by summing the conservation laws for Lagrangian L_β along with generators $U = t_\beta \partial_{x_\beta} + \frac{1}{c^2} x_\beta \partial_{t_\beta}$ and $U = t_{\bar{\beta}} \partial_{x_{\bar{\beta}}} + \frac{1}{c^2} x_{\bar{\beta}} \partial_{t_{\bar{\beta}}}$ in the two-time case then taking $t_\beta \to \tau$.

Equations (33) have one dilatation symmetry and two Lagrangians, so two linearly independent conservation laws may be found as opposed to four in the case of the two-time wave equations. The generator

$$U = -\frac{5}{2}\Gamma \partial_\Gamma - \frac{5}{2}\Gamma^* \partial_{\Gamma^*} + \sum_i \chi^i \partial_{\chi^i}, \tag{81}$$

which is a linear combination the dilatation generator and the linearity generator, corresponds to a variational symmetry. This generator along with Lagrangian L_β corresponds to a conservation law in the form of Eq. (14) where

$$\mathbf{P} = \left(\nabla_\beta \Gamma^* - \frac{1}{c^2}\frac{\partial \Gamma^*}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\left(\frac{5}{2}\Gamma + \sum_i \chi^i \frac{\partial \Gamma}{\partial \chi^i}\right) + \left(\nabla_\beta \Gamma - \frac{1}{c^2}\frac{\partial \Gamma}{\partial t_\beta}\mathbf{a}_{t_\beta}\right)\left(\frac{5}{2}\Gamma^* + \sum_i \chi^i \frac{\partial \Gamma^*}{\partial \chi^i}\right) + L_\beta \left(\sum_i \chi^i \mathbf{a}_i\right). \tag{82}$$

Equations (33) do not contain inversion symmetries, so no corresponding conservation laws can be found. For each infinitesimal generator U of the inversion symmetries, the corresponding conservation law in the two-time case may be simplified to the form

$$\sum_i \frac{\partial \left(P_i - B_i\right)}{\partial \chi^i} = 0 = \left(\nabla_\beta^2 \Gamma - \frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t_\beta^2}\right)\left(U\Gamma^*\right) + \left(\nabla_\beta^2 \Gamma^* - \frac{1}{c^2}\frac{\partial^2 \Gamma^*}{\partial t_\beta^2}\right)\left(U\Gamma\right) \tag{83}$$

where $U\Gamma$ represents the cross-correlation upon the symmetry transformation. In order for such a conservation law, which appears for the nonstationary fields, to also hold for the stationary fields, both Γ and $U\Gamma$ must be independent of $T = \frac{t_1+t_2}{2}$. It may be seen that this can only be true for $\Gamma = 0$, so the conservation laws arising from the inversion symmetry in the two-time case become trivial for stationary fields.

4. Conclusion

The symmetries and conservation laws for the Wolf equations are studied systematically here by the Lie method and Noether's theorem. All of the symmetries of the deterministic case are found in the two-time stochastic case but for both sets of coordinates. The stationary stochastic case is different in that some symmetries are eliminated. The two-time stochastic case contains generalizations of all of the conservation laws of the deterministic case, and the deterministic limit may be taken to obtain the deterministic conservation laws as a special case. For example, the four energy conservation laws of the two-time case reduce to the one conservation law for the deterministic case in the deterministic limit. In the stationary case some conservation laws for the two-time case reduce or coalesce to corresponding laws for the stationary case. For example, the four energy conservation laws in the two-time case become two conservation laws for the stationary case. In other instances, conservation laws are simply eliminated in going from the two-time case to the stationary case, such as the conservation law resulting from the inversion symmetries.

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